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Isospin violation in $\tau \to 3\pi\nu_\tau$

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Abstract. Isospin violating signals in the $\tau^- \to (3\pi)^-\nu_\tau$ decay mode are discussed. For the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ decay mode, isospin violation arises from the vector current contribution in the $\tau^- \to \omega\pi^-\nu_\tau$ decay with the subsequent isospin violating ω decay into $\pi^+\pi^-$. We demonstrate that such effects may be observed in presently available data through the measurement of the interference effects of these vector current contributions with the dominating axial vector current, *i.e.* through a measurement of the structure functions W_F, W_G, W_H and W_I. In the case of the $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ decay mode, a vector current contribution is generated by $\eta\pi^0$ mixing in the decay chain $\tau^- \to \eta\rho^-\nu_\tau \to \pi^0\pi^0\pi^-\nu_\tau$. We find that this effect is rather small, the magnitude of the associated interference terms being too low for present statistics.

I Introduction

Isospin rotations have been successfully used in τ decays into an even number of final state pions to relate the vector current to the corresponding cross sections measured in electron positron collisions [1,2]. In the case of the two pion mode, the τ decay rate has been measured with a relative error below one percent which is of the size of possible isospin violating effects. Isospin symmetry relations are also very useful to relate various decay amplitudes in $3\pi\nu_\tau$, $KK\pi\nu_\tau$ and $K\pi\pi\nu_\tau$ final states [3–5].

Isospin violation effects in the decay $\tau^- \to \omega\pi^-\nu_\tau$ have been discussed in [6]. Such signals could be revealed by an analysis of the angular distribution in the $\omega\pi^-$ system. Another interesting isospin violating process is provided by the decay $\tau^- \to \eta\pi^-\nu_\tau$. The different theoretical predictions for the branching ratio [7,8] are still one order of magnitude smaller than the actual experimental upper limit [9].

In this article we will concentrate on possible isospin violating effects in the $\tau \to 3\pi\nu_\tau$ decay mode. Although the theoretical uncertainties in this decay mode are fairly large, observations of small isospin violating effects (below 1% to the rate) might be possible with presently available statistics. The sensitivity to such small effects is provided by an analysis of angular distributions. The relevant information is encoded in structure functions [10,11] which allow to reconstruct the form factors in the dominating axial current and in the small isospin violating vector current contributions. In particular the interference effects between the vector and the axial vector amplitudes, given by the structure functions W_F, W_G, W_H, W_I allow for such a measurement. Any nonvanishing contribution to these structure functions would be a clear signal of isospin violation in the three pion decay mode of the τ. After specifying the isospin violating vector form factor, we will present numerical predictions for these structure functions including the full dependence on the resonance structure. We also analyze Dalitz distributions for the purely axial vector structure functions. Such distributions, in particular for the structure function W_D, are fairly sensitive to the details of the ρ sub-resonance implementation in the underlying models.

A branching fraction of 0.6% in the $\tau \to 3\pi\nu_\tau$ mode due to isospin violation has been reported by the ARGUS collaboration [12]. Their analysis is based on a study of eight different contributions to the amplitude. Unfortunately the relevant interference terms with the isospin conserving part of the amplitude cannot be traced out unambiguously from that work.

The paper is organized as follows: The general structure of the decay amplitude and the structure function formalism in the three meson decay mode is briefly summarized in Sect. II and a particular choice for the form factors in the axial vector current, the Kühn–Santamaria model [1], is specified in Sect. III. Isospin violating contributions to an additional vector current form factor will be discussed in Sect. IV (Sect.V) for the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ ($\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$) decay mode. The relevant hadronic matrix elements are determined in the vector meson dominance model. We obtain as a by-product from the decay $\tau^- \to \eta\pi^0\pi^-\nu_\tau$ a new parametrization for the transition of the ρ resonance into three pseudoscalar mesons which is also needed as an input *e.g.* for the τ decay into $KK\pi\nu_\tau$ final states. Finally, isospin violating signals induced by the vector current form factor are discussed in Sect. VI.

II Three meson decay modes: form factors and structure functions

The matrix element $\mathcal{M}$ for the semi-leptonic τ decay into three mesons h_1, h_2, h_3

$$\tau(l,s) \to \nu_\tau(l',s') + h_1(q_1,m_1) + h_2(q_2,m_2) + h_3(q_3,m_3) \tag{1}$$

can be expressed in the following form:

$$\begin{aligned}&\mathcal{M}(\tau \to \nu_\tau\ h_1h_2h_3)\\ &= \frac{G_F}{\sqrt{2}} \binom{\cos\theta_c}{\sin\theta_c}\ \bar{u}(l',s')\gamma_\mu(1-\gamma_5)u(l,s)\ J^\mu.\end{aligned} \tag{2}$$

In (2) G_F denotes the Fermi-coupling constant and θ_c is the Cabibbo angle. The hadronic current

$$J^\mu(q_1,q_2,q_3) = \langle h_1(q_1)h_2(q_2)h_3(q_3))|V^\mu(0) - A^\mu(0)|0\rangle \tag{3}$$

is characterized by four independent form factors F_1, F_2, F_3, F_4 [10]. These form factors are in general functions of $s_1 = (q_2+q_3)^2, s_2 = (q_1+q_3)^2$, $s_3 = (q_1+q_2)^2$ and Q^2, which is conveniently chosen as an additional variable.

$$J^\mu(q_1,q_2,q_3) = V_1^\mu\, F_1 + V_2^\mu\, F_2 + i\, V_3^\mu\, F_3 + V_4^\mu\, F_4 \tag{4}$$

with

$$\begin{aligned} V_1^\mu &= (q_1-q_3)_\nu\, T^{\mu\nu}\\ V_2^\mu &= (q_2-q_3)_\nu\, T^{\mu\nu}\\ V_3^\mu &= \epsilon^{\mu\alpha\beta\gamma} q_{1\,\alpha} q_{2\,\beta} q_{3\,\gamma}\\ V_4^\mu &= q_1^\mu + q_2^\mu + q_3^\mu = Q^\mu.\end{aligned} \tag{5}$$

$T^{\mu\nu}$ denotes the transverse projector

$$T_{\mu\nu} = g_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2}\,. \tag{6}$$

F_1 and F_2 determine the spin one component of the axial vector current induced amplitude, F_4 the spin zero part which is given by the matrix element of the divergence of the axial vector current. The vector current induced amplitude is responsible for the form factor F_3. All form factors may contribute in the general three meson case [13,14,3]. G-parity conservation and PCAC in the three pion decay mode implies $F_3 = F_4 = 0$. However, isospin violation is expected to give a nonvanishing contribution to F_3. Such contributions will be studied in the last three sections of this paper.

The three meson decay in (1) is most easily analyzed in the hadronic rest frame $\mathbf{q_1} + \mathbf{q_2} + \mathbf{q_3} = 0$. The orientation of the hadronic system is in general characterized by three Euler angles (α, β and γ) as introduced in [10,11]. Of particular interest are the distributions of the normal to the Dalitz plane and the distributions around this normal. Performing the analysis in the hadronic rest frame has the advantage that the product of the hadronic tensor ($H^{\mu\nu} = J^\mu (J^\nu)^\dagger$) and the leptonic tensor reduces to a sum $L^{\mu\nu}H_{\mu\nu} = \sum_X \bar{L}_X W_X$. The leptonic factors $\bar{L}_X$ factorize the dependence on the Euler angles. For the definition of these angles and the explicit dependence of the coefficients $\bar{L}_X$ on α, β and γ see [10]. The (in general 16) hadronic structure functions W_X correspond to 16 density matrix elements for a hadronic system in a spin one and spin zero state (nine of them originate from a pure spin one state and the remaining originate from a pure spin zero state or from interference terms between spin zero and spin one). These structure functions contain the dynamics of the three meson decay and depend only on the form factors F_i and on the hadronic invariants Q^2 and the Dalitz plot variables s_i. The scalar contribution is expected to be small [15] for all three meson final states and will be neglected in the subsequent discussion of this paper[1]. Instead of the 16 real structure functions which characterize the general hadronic tensor $H^{\mu\nu}$ one thus deals only with nine functions W_X. These nine structure functions can be divided in four functions which arise only from the axial vector current ($W_{A,C,D,E}$), one from the vector current (W_B) and the remaining four from the interference of the axial vector and vector current ($W_{F,G,H,I}$). The latter will be of particular importance in the subsequent discussion.

The dependence of the structure functions on the form factors F_i reads [10]:
Axial vector structure functions:

$$\begin{aligned} W_A &= \quad (x_1^2 + x_3^2)\,|F_1|^2 + (x_2^2 + x_3^2)\,|F_2|^2\\ &\quad +2(x_1x_2 - x_3^2)\,\mathrm{Re}\,(F_1F_2^*)\\ W_C &= \quad (x_1^2 - x_3^2)\,|F_1|^2 + (x_2^2 - x_3^2)\,|F_2|^2\\ &\quad +2(x_1x_2 + x_3^2)\,\mathrm{Re}\,(F_1F_2^*)\\ W_D &= \quad 2\left[x_1x_3\,|F_1|^2 - x_2x_3\,|F_2|^2\right.\\ &\quad \left.+x_3(x_2 - x_1)\,\mathrm{Re}\,(F_1F_2^*)\right]\\ W_E &= -2x_3(x_1 + x_2)\,\mathrm{Im}\,(F_1F_2^*)\end{aligned} \tag{7}$$

Vector structure function:

$$W_B = \quad x_4^2|F_3|^2 \tag{8}$$

Axial vector–vector interference structure functions:

$$\begin{aligned} W_F &= \quad 2x_4\left[x_1\,\mathrm{Im}\,(F_1F_3^*) + x_2\,\mathrm{Im}\,(F_2F_3^*)\right]\\ W_G &= -2x_4\left[x_1\,\mathrm{Re}\,(F_1F_3^*) + x_2\,\mathrm{Re}\,(F_2F_3^*)\right]]\\ W_H &= \quad 2x_3x_4\left[\mathrm{Im}\,(F_1F_3^*) - \mathrm{Im}\,(F_2F_3^*)\right]\\ W_I &= -2x_3x_4\left[\mathrm{Re}\,(F_1F_3^*) - \mathrm{Re}\,(F_2F_3^*)\right]\end{aligned} \tag{9}$$

The variables x_i are defined by

$$\begin{aligned} x_1 &= V_1^x = q_1^x - q_3^x\\ x_2 &= V_2^x = q_2^x - q_3^x\\ x_3 &= V_1^y = q_1^y = -q_2^y\\ x_4 &= V_3^z = \sqrt{Q^2}x_3q_3^x\end{aligned} \tag{10}$$

[1] Using an ansatz for a scalar contribution in the $3\pi\nu_\tau$ decay mode as specified in [10], U. Müller constrained such a contribution in the branching ratio to be less than 0.84 % by analyzing the spin-zero-spin-one structure functions with 1994 OPAL data [16]. Note that a possible scalar contribution would not contribute to the vector-axial vector interference structure functions in (9) which are important for an observation of isospin violating effects

where q_i^x (q_i^y) denotes the x (y) component of the momentum of meson i in the hadronic rest frame. They can easily be expressed in terms of s_1, s_2 and s_3 [10]. $W_A(Q^2, s_i)$ and $W_B(Q^2, s_i)$ govern the rate and the distributions in the Dalitz plot through

$$\Gamma(\tau \to 3h\nu_\tau) = \frac{G^2}{12m_\tau} \binom{\cos\theta_c}{\sin\theta_c}^2 \frac{1}{(4\pi)^5} \times \int \frac{dQ^2}{Q^4} ds_1 ds_2 \, (m_\tau^2 - Q^2)^2 \left(1 + \frac{2Q^2}{m_\tau^2}\right) \; (W_A + W_B) \tag{11}$$

The remaining structure functions determine the angular distribution. All of them can be determined by a measurement of the β and γ dependence even without reconstructing the τ rest frame.

III Axial vector current contribution to $\tau^- \to (3\pi)^-\nu_\tau$

τ decays into three pions are dominated by the axial vector current which allows for significant simplifications: G-parity implies $F_3 = 0$, Bose symmetry relates F_1 and F_2 through $F_2(s_1, s_2, Q^2) = F_1(s_2, s_1, Q^2)$ and PCAC leads to $F_4 = 0$. Note that the structure functions $W_{B,F,G,H,I}$ in (8,9) vanish for $F_3 = 0$.

The two like-sign pions in $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ and $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ are labeled such that $|\mathbf{p}_2| > |\mathbf{p}_1|$ and p_3 refers to the unlike-sign pion. The normalization of the form factors F_1 and F_2 for the three pion decay mode is determined in the chiral limit[2] [17],

$$F_1 = F_2 = i\frac{2\sqrt{2}}{3f_\pi}, \qquad f_\pi = 93 \text{ MeV} \tag{12}$$

For large Q^2, s_1 and s_2 these form factors are modulated by resonances in the 3π and 2π channel. Following the ansatz of Kühn and Santamaria [1], one has

$$F_1(Q^2, s_2) = i\frac{2\sqrt{2}}{3f_\pi} \; BW_{a_1}(Q^2)\, T_\rho^{(2m)}(s_2) \tag{13}$$

$$F_2(Q^2, s_1) = i\frac{2\sqrt{2}}{3f_\pi} \; BW_{a_1}(Q^2)\, T_\rho^{(2m)}(s_1) \tag{14}$$

The Breit–Wigner functions $\mathrm{BW}_X(s)$ are parametrized including energy dependent widths,

$$\mathrm{BW}_X(s) = \frac{m_X^2}{m_X^2 - s - i\sqrt{s}\Gamma_X(s)} \quad , \quad \mathrm{BW}_X(0) = 1 \tag{15}$$

For the a_1 we have in particular

$$\Gamma_{a_1}(s) = \frac{m_{a_1}}{\sqrt{s}}\, \Gamma_{a_1} \frac{g(s)}{g(m_{a_1}^2)} \; , \quad m_{a_1} = 1.251 \text{ GeV} \; , \quad \Gamma_{a_1} = 0.475 \text{ GeV} \tag{16}$$

where the function $g(s)$ has been calculated in [1] and is derived from the observation, that the axial vector resonance a_1 decays predominately into three pions.

[2] We use the Condon-Shortley phase conventions

The superscript $(2m)$ in the ρ form factor $T_\rho^{(2m)}(s)$ denotes the subsequent decay into two pions. In the parametrization of $T_\rho^{(2m)}(s)$ one allows for a contribution of the first excitation ρ',

$$T_\rho^{(2m)}(s) = \frac{1}{1+\beta_\rho}\left[\mathrm{BW}_\rho(s) + \beta_\rho\, \mathrm{BW}_{\rho'}(s)\right] , \tag{17}$$

with the energy dependent width

$$\Gamma_\rho(s) = \Gamma_\rho \, \frac{m_\rho^2}{s} \left(\frac{s - 4m_\pi^2}{m_\rho^2 - 4m_\pi^2}\right)^{3/2} \tag{18}$$

and similarly for the ρ'. The parameters are given by

$$\begin{aligned} \beta_\rho &= -0.145 \, , \\ m_\rho &= 0.773\,\text{GeV} \, , \quad \Gamma_\rho = 0.145\,\text{GeV} \, , \\ m_{\rho'} &= 1.370\,\text{GeV} \, , \quad \Gamma_{\rho'} = 0.510\,\text{GeV} \, . \end{aligned} \tag{19}$$

which have been determined from $e^+e^- \to \pi^+\pi^-$ in [1]. Predictions for the $(s_1 - s_2)$-integrated structure functions $w_X(Q^2) = \int ds_1 ds_2 W_X(Q^2, s_1, s_2)$ for $X = A, C, D, E$ based on this model are in good agreement with data [18]. The invariant 3π and 2π mass distributions for the four integrated nonvanishing structure functions $W_A, W_C, sign\,(s_1 - s_2)W_D, sign(s_1 - s_2)W_E$ in Fig. 1 reveal the importance of the a_1 (solid) and ρ (dashed) resonances. The $\sqrt{s_3}$ distribution (dotted line) is then fixed by phase space restrictions and the $\sqrt{Q^2}$ and $\sqrt{s_{1,2}}$ distributions through $s_3 = Q^2 - s_1 - s_2 + 3m_\pi^2$. The structure functions W_D and W_E are combined with an energy ordering $sign(s_1 - s_2)$ to account for Bose symmetry. The $\sqrt{s_{1,2}}$ distributions of W_A, W_C and $sign(s_1 - s_2)W_F$ have a clear peak around the ρ resonance, whereas $sign(s_1 - s_2)W_D(\sqrt{s_{1,2}})$ has a surprisingly different behaviour in the Kühn-Santamaria model. The distribution shows a relatively wide peak around $\sqrt{s_{1,2}} = 0.5$ GeV and only a much smaller additional peak around the ρ mass. In contrast, the $\sqrt{s_{1,2}}$ distribution for $sign(s_1 - s_2)W_D$ based on the model in [19] has its maximum around the ρ mass and only a small additional peak around $\sqrt{s_{1,2}} = 0.5$. An experimental confirmation of the predictions for the $\sqrt{s_{1,2}}$ distributions in the axial vector structure functions shown in Fig. 1 and in particular in $sign(s_1 - s_2)W_D$ would be a good test of the details in the ρ resonance structure in the Kühn Santamaria model which we use for the two axial vector form factors F_1 and F_2.

IV Vector current contribution to $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$

More detailed studies, such as testing the magnitude of amplitudes induced by F_3 through isospin violation, are possible and will be discussed in the following. Since they affect the angular distributions through interference terms between the (small) contribution from F_3 with the large contributions from F_1 and F_2 (see (9)), they should be accessible in measurements of the structure functions

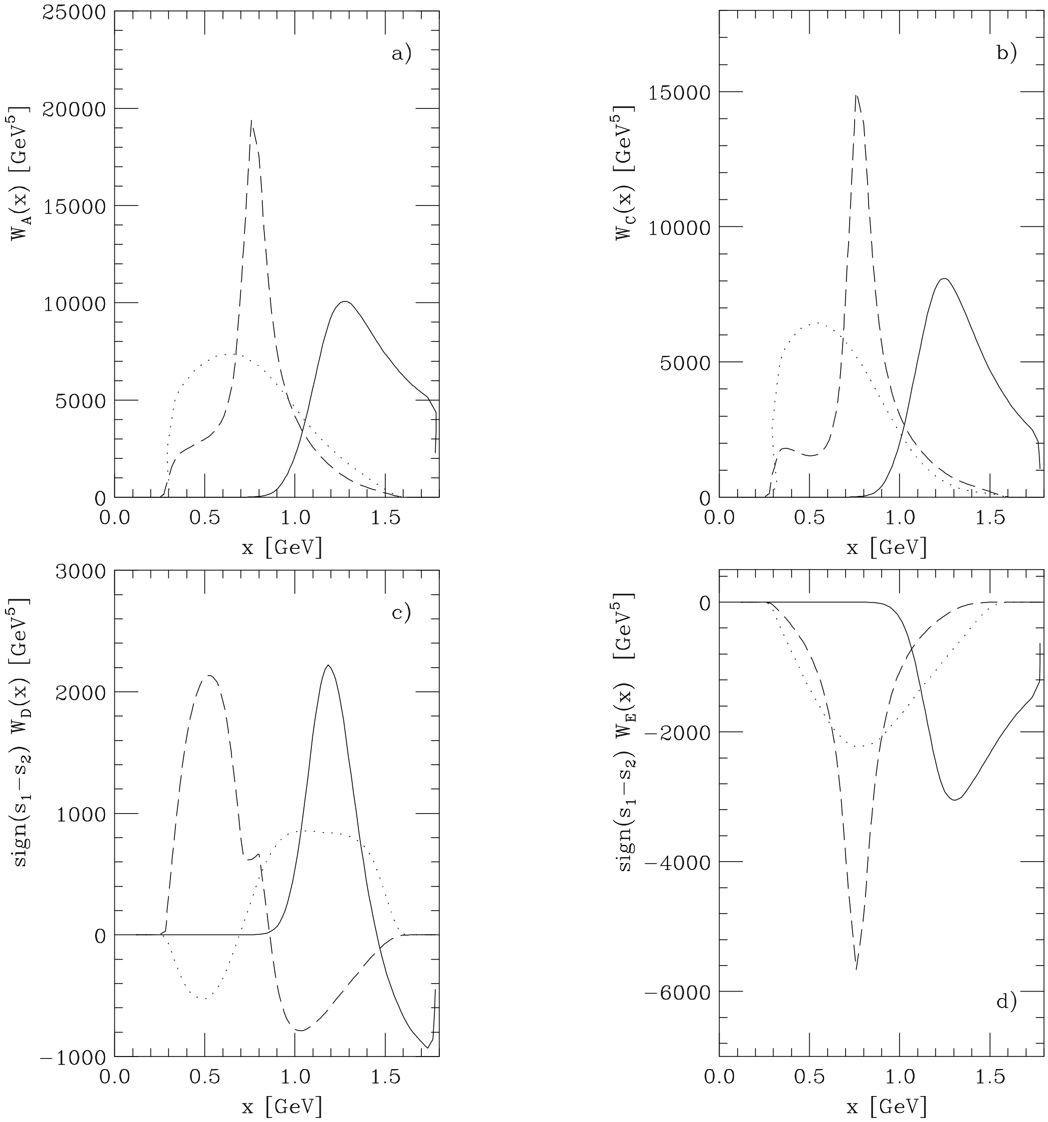

Fig. 1. Invariant mass distributions $x = \sqrt{Q^2} = m(\pi^-\pi^-\pi^+)$ (*solid*), $x = \sqrt{s_{1,2}} = m(\pi^+\pi^-)$ (*dashed*) and $x = \sqrt{s_3} = m(\pi^-\pi^-)$ (*dotted*) of the structure functions W_A **a**, W_C **b**, $sign(s_1-s_2)W_D$ **c**, $sign(s_1-s_2)W_E$ **d** in the $\tau^- \to (3\pi)^-\nu_\tau$ decay mode

$W_{F,G,H,I}$, already with the statistics of ongoing experiments.

A small vector current contribution ($\sim F_3$ in (4)) to the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ mode is expected to arise from the $\tau^- \to \omega\pi^-\nu_\tau$ decay with the subsequent isospin violating ω decay into $\pi^+\pi^-$. G-parity requires that the $\omega\pi^-$ system is in a 1^- state and hence the $\tau^- \to \omega\pi^-\nu_\tau$ decay can only proceed via a vector current. The hadronic matrix element is determined through [3,20] (for another approach see [21])

$$\begin{aligned}
\langle\omega(\tilde{q}_1,\lambda)\pi^-(\tilde{q}_2)|V^\mu(0)|0\rangle &= i\,\epsilon^{\mu\alpha\beta\gamma}\,\varepsilon^*_\alpha(\tilde{q}_1,\lambda)\,\tilde{q}_{1\,\beta}\tilde{q}_{2\,\gamma} \\
&\quad\times\, F_V^{(\omega\pi)}(Q^2) \\
F_V^{(\omega\pi)}(Q^2) &= i\,\frac{f_{\rho^-}\,g_{\rho\omega\pi}}{m_\rho^2}\,T_\rho^{(4m)}(Q^2) \qquad (20) \\
Q^2 &= (\tilde{q}_1+\tilde{q}_2)^2
\end{aligned}$$

where V^μ is the vector part of the weak current. Note that we have fixed the sign of $F_V^{(\omega\pi)}(0)$ from $\pi^0 \to \gamma\gamma$. f_{ρ^-} is the coupling of the charged $\rho^\pm$ to the gauge boson $W^\pm$ and is related to the $\rho^0\gamma$ coupling f_ρ, $g_{\rho\omega\pi}$ is measured in the decays $\omega \to \pi^0\gamma$ and $\omega \to \pi^+\pi^-\pi^0$ [22], respectively,

$$f_{\rho^-} = \sqrt{2}\,f_\rho \;\simeq\; 0.17\ \text{GeV}^2 \qquad (21)$$

$$g_{\rho\omega\pi} = \begin{cases} 11.7 \pm 0.4\ \text{GeV}^{-1} & \text{from } \Gamma\left(\omega \to \pi^0\gamma\right) \\ 15.0 \pm 0.1\ \text{GeV}^{-1} & \text{from } \Gamma\left(\omega \to \pi^+\pi^-\pi^0\right) \end{cases}$$

The ρ-meson and its radial excitations are possible resonance candidates for the vector form factor $T_\rho^{(4m)}(s)$, where the superscript $(4m)$ refers to the (anomalous) VMD decay chain $\rho \to \omega\pi \to 4\pi$. The admixture of the radial excitations in $T_\rho^{(4m)}(s)$ is expected to differ from the corresponding ρ form factor $T_\rho^{(2m)}(s)$ with a dominant two pion decay in (17). Here we allow for an admixture of the

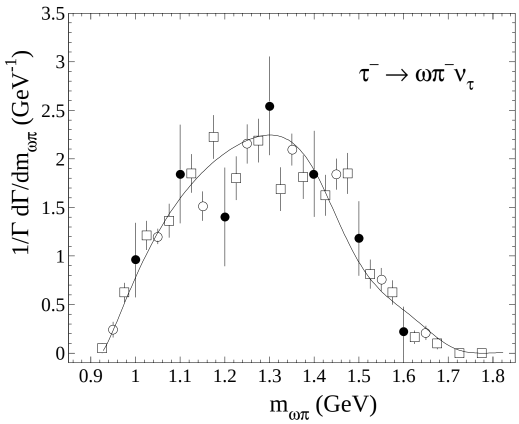

Fig. 2. The invariant $\omega\pi$ mass spectrum from $\tau^- \to \omega\pi^-\nu_\tau$ measured by ARGUS [23] (*filled circles*), by CLEO [24] (*opened circles*) and by ALEPH [25] (*opened squares*). The *solid line* shows the fit result to (20,23)

ρ' and the ρ'' via

$$T_\rho^{(4m)}(s) = \frac{1}{1+\lambda+\kappa}\Big[\mathrm{BW}_\rho(s) + \lambda \mathrm{BW}_{\rho'}(s) + \kappa \mathrm{BW}_{\rho''}(s)\Big] \tag{22}$$

where we fix the parameters to the PDG [22] values, which yields

$$\begin{array}{l} m_\rho = 0.773 \text{ GeV}\,, \;\; \Gamma_\rho = 0.145 \text{ GeV} \\ m_{\rho'} = 1.465 \text{ GeV}\,, \; \Gamma_{\rho'} = 0.310 \text{ GeV} \\ m_{\rho''} = 1.70 \text{ GeV}\,, \;\; \Gamma_{\rho''} = 0.235 \text{ GeV}\,. \end{array} \tag{23}$$

The parameters λ and κ are obtained from a fit to the normalized invariant mass spectrum of $\tau^- \to \omega\pi^-\nu_\tau$ data [23–25], see Fig. 2,

$$\begin{aligned} \lambda &= -0.054 \pm 0.012 \\ \kappa &= -0.036 \pm 0.004 \\ \chi^2/\text{d.o.f.} &= 53.3/31 \quad . \end{aligned} \tag{24}$$

Note that the errors should be taken as an educated guess only, since we fit to the published data with the correlation matrices to be diagonal, and the mass and width parameters in (23) are considered exact values. The values in (24) lead to the following branching ratios, depending strongly on the ω decay channel from which one extracts $g_{\rho\omega\pi}$,

$$\mathcal{B}\left(\tau^- \to \omega\pi^-\nu_\tau\right) = \left\{ \begin{array}{ll} (0.98 \pm 0.21)\ \% & g_{\rho\omega\pi} = 11.7 \pm 0.4 \ \text{GeV}^{-1} \\ (1.61 \pm 0.23)\ \% & g_{\rho\omega\pi} = 15.0 \pm 0.1 \ \text{GeV}^{-1} \end{array}\right. \tag{25}$$

The errors in the branching ratios are dominated by the errors in λ and κ in (24). A comparison to the measured experimental branching ratios shows that small values for $g_{\rho\omega\pi}$ are excluded,

$$\mathcal{B}_{exp.}\left(\tau^- \to \omega\pi^-\nu_\tau\right) = (1.92 \pm 0.08)\ \% \tag{26}$$

where we combined the measured branching fractions from CLEO [24] and ALEPH [25] . Thus we will put $g_{\rho\omega\pi} = 15.0 \text{ GeV}^{-1}$ in the following, keeping in mind that the measured τ decay rate would even require a higher value of $g_{\rho\omega\pi}$.

The transition $\omega \to \pi^+\pi^-$ is assumed to proceed through $\rho^0\omega$ mixing and is written in the form

$$\langle \pi^+(k_1)\pi^-(k_2)|\mathcal{T}|\omega(k,\lambda)\rangle = \frac{\theta_{\rho\omega}g_{\rho\pi\pi}}{m_\rho^2}\mathrm{BW}_\rho(k^2)(k_1-k_2)_\mu\,\varepsilon^\mu(k,\lambda) \tag{27}$$

where $g_{\rho\pi\pi}$ is related to the decay $\rho^0 \to \pi^+\pi^-$, and the $\rho^0\omega$ mixing parameter $\theta_{\rho\omega}$ is measured in $e^+e^- \to \pi^+\pi^-$ experiments [26],

$$g_{\rho\pi\pi} = 6.08, \quad \theta_{\rho\omega} = (-3.97 \pm 0.20) \times 10^{-3} \text{ GeV}^2\,. \tag{28}$$

Combining the amplitudes in (20, 27) one obtains for the three pion decay mode after summation over the polarization λ of the intermediate ω state,

$$\langle \pi^-\pi^-\pi^+|V^\mu|0\rangle = \sum_\lambda \langle \pi^+\pi^-|\mathcal{T}|\omega(p,\lambda)\rangle \times \langle \omega(p,\lambda)\,\pi^-|V^\mu|0\rangle\ \frac{-1}{m_\omega^2}\mathrm{BW}_\omega(s) \tag{29}$$

where $s = p^2$ is the momentum transfer and the width in $\mathrm{BW}_\omega(s)$ is chosen to be energy independent due to its smallness,

$$BW_\omega(s) = \frac{m_\omega^2}{m_\omega^2 - s - im_\omega\Gamma_\omega}, \qquad M_\omega = 0.782 \text{ GeV}, \quad \Gamma_\omega = 8.4 \text{ MeV} \tag{30}$$

With the identity $\sum_\lambda \varepsilon_\mu(p,\lambda)\varepsilon^*_\nu(p,\lambda) = -g^{\mu\nu} + p^\mu p^\nu/m^2$ we find the following parametrization of the form factor F_3,

$$\begin{aligned} &\langle \pi^-(q_1)\pi^-(q_2)\pi^+(q_3)|V^\mu(0)|0\rangle \\ &\quad = i\,\epsilon^{\mu\alpha\beta\gamma}\,q_{1\,\alpha}q_{2\,\beta}q_{3\,\gamma}\,F_3(s_1,s_2,Q^2) \\ &F_3(s_1,s_2,Q^2) \\ &\quad = -2\frac{\theta_{\rho\omega}g_{\rho\pi\pi}}{m_\rho^2 m_\omega^2}F_V^{(\omega\pi)}(Q^2) \\ &\qquad \times\left[\mathrm{BW}_\omega(s_1)\mathrm{BW}_\rho(s_1) - \mathrm{BW}_\omega(s_2)\mathrm{BW}_\rho(s_2)\right]\,. \end{aligned} \tag{31}$$

V Vector current contribution to $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$

In the case of the $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ decay mode we assume that the vector current contribution is generated by $\eta\pi^0$ mixing in the decay chain $\tau^- \to \eta\rho^-\nu_\tau \to \pi^0\pi^0\pi^-\nu_\tau$.

The $\tau^- \to \eta\pi^0\pi^-\nu_\tau$ decay is allowed to proceed in the Standard Model via a vector current induced by the Wess-Zumino anomaly part in the Lagrangian [27]. A normalization of the form factor $F_3^{(\eta\pi\pi)}$ is fixed in the chiral limit and a parametrization of $F_3^{(\eta\pi\pi)}$ reads [7,14,28,29]

$$\langle \eta(q_1)\pi^0(q_2)\pi^-(q_3)|V^\mu(0)|0\rangle$$

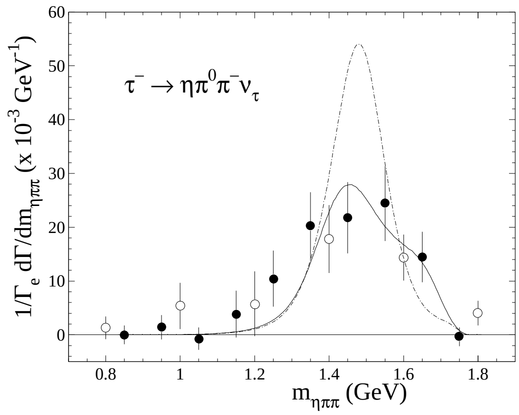

Fig. 3. The $\eta\pi\pi$ mass spectrum from $\tau^- \to \eta\pi^0\pi^-\nu_\tau$ measured by CLEO [31] (*filled circles*) and by ALEPH [25] (*open circles*) normalized to $\Gamma_e = \Gamma(\tau^- \to e^-\overline{\nu}_e\nu_\tau)$. The *solid line* shows the fit result to (33), the *dashed line* represents the $\eta\pi\pi$ mass spectrum obtained from $e^+e^- \to \eta\pi\pi$ data [28,30]

$$= i\,\epsilon^{\mu\alpha\beta\gamma}\, q_{1\,\alpha} q_{2\,\beta} q_{3\,\gamma}\, F_3^{(\eta\pi\pi)}(s_1, Q^2)$$
$$F_3^{(\eta\pi\pi)}(s_1, Q^2) = i\,\frac{\sqrt{6}}{12\pi^2 f_\pi^3}\, T_\rho^{(3m)}(Q^2)\, T_\rho^{(2m)}(s_1)\ . \qquad (32)$$

The form and parameters of $T_\rho^{(2m)}(s)$ are given in (17,19). For the form factor $T_\rho^{(3m)}(s)$ the superscript $(3m)$ implies the anomalous transition $\rho \to \eta\rho \to \eta\pi\pi$. In [3, 14,28], a form for $T_\rho^{(3m)}(s)$ including ρ, ρ' and ρ'' was used, which has been obtained from a fit to (fairly poor) $e^+e^- \to \eta\pi\pi$ data [28,30]. However, new measurements for $\tau^- \to \eta\pi^0\pi^-\nu_\tau$ have become available allowing now for a direct determination of $T_\rho^{(3m)}(s)$ in τ decays [25,31]. A direct fit to the differential decay rate for $\tau^- \to \eta\pi^0\pi^-\nu_\tau$ normalized to $\Gamma_e = \Gamma(\tau^- \to e^-\overline{\nu}_e\nu_\tau)$ as shown in Fig. 3 (solid line) yields for the coefficients ξ and σ:

$$T_\rho^{(3m)}(s) = \frac{1}{1+\xi+\sigma}\left[\mathrm{BW}_\rho(s) + \xi\mathrm{BW}_{\rho'}(s) + \sigma\mathrm{BW}_{\rho''}(s)\right]$$
$$\xi = -0.22 \pm 0.03$$
$$\sigma = -0.10 \pm 0.01$$
$$\chi^2/\text{d.o.f.} = 11.0/14\ . \qquad (33)$$

where the masses and widths of the resonances are given in (23). Again the errors have to be considered educated ones, see the remark in Sect. III. The branching fraction that we obtain is compatible with the measured decay rate,

$$\mathcal{B}\left(\tau^- \to \eta\pi^0\pi^-\nu_\tau\right) = (0.14 \pm 0.05)\,\%$$
$$\mathcal{B}_{exp.}\left(\tau^- \to \eta\pi^0\pi^-\nu_\tau\right) = (0.17 \pm 0.03)\,\%\ . \qquad (34)$$

where we give the weighted average of the experimental branching fractions from CLEO [31] and ALEPH [25]. Thus the invariant mass distribution and the decay rate for the $\tau^- \to \eta\pi^0\pi^-\nu_\tau$ decay mode are well described by these parameters. On the other hand we found that the $\eta\pi^0\pi^-$ invariant mass spectrum in Fig. 3 is only poorly described by the $T_\rho^{(3m)}(s)$ parametrization based on the $e^+e^- \to \eta\pi\pi$ data (dashed line in Fig. 3).

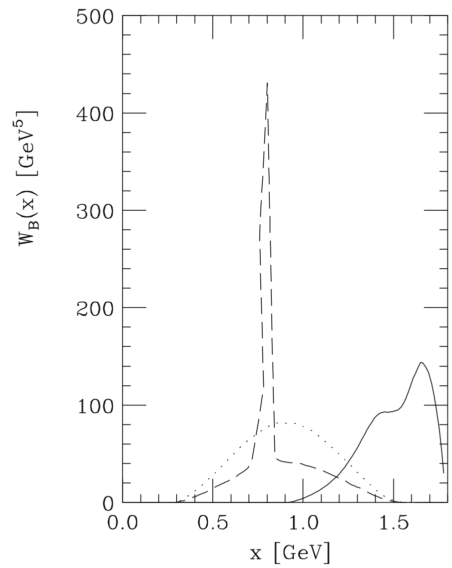

Fig. 4. Invariant mass distributions $x = \sqrt{Q^2} = m(\pi^-\pi^-\pi^+)$ (*solid*), $x = \sqrt{s_1} = \sqrt{s_2} = m(\pi^+\pi^-)$ (*dashed*) and $x = \sqrt{s_3} = m(\pi^-\pi^-)$ (*dotted*) of the structure function W_B in the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ decay mode

For the isospin violating form factor in the three pion decay we deduce the form

$$\langle \pi^0(q_1)\pi^0(q_2)\pi^-(q_3)|V^\mu(0)|0\rangle = i\,\epsilon^{\mu\alpha\beta\gamma}\, q_{1\,\alpha} q_{2\,\beta} q_{3\,\gamma}\, F_3(s_1, s_2, Q^2)$$
$$F_3(s_1, s_2, Q^2) = \varepsilon\left[F_3^{(\eta\pi\pi)}(s_1, Q^2) - F_3^{(\eta\pi\pi)}(s_2, Q^2)\right]\ , \qquad (35)$$

where an estimate of the $\eta\pi^0$ mixing parameter ε is given by [32]

$$\varepsilon = (1.05 \pm 0.07) \times 10^{-2}\ . \qquad (36)$$

An additional decay channel would be induced by $\eta\eta'$ mixing with a subsequent $\eta'\pi^0$ transition. Experimentally, the decay $\tau^- \to \eta'\pi^0\pi^-\nu_\tau$ has not been observed [33] and thus a reliable parametrization of the associated form factor $F_3^{(\eta'\pi\pi)}$ is missing. We therefore neglect possible contributions from the η' as an intermediate state.

VI Numerical results

After having fixed our model for the isospin violating vector current contributions to the three pion decay mode, we next discuss numerical effects of this contribution to the decay widths and in particular to the structure functions $W_B, sign(s_1-s_2)W_F, sign(s_1-s_2)W_G, W_H$ and W_I. The structure functions W_F and W_G are again combined with an energy ordering $sign(s_1-s_2)$ to account for Bose symmetry.

Let us start with the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ decay mode. Fig. 4 shows the resonance structure of the pure vector

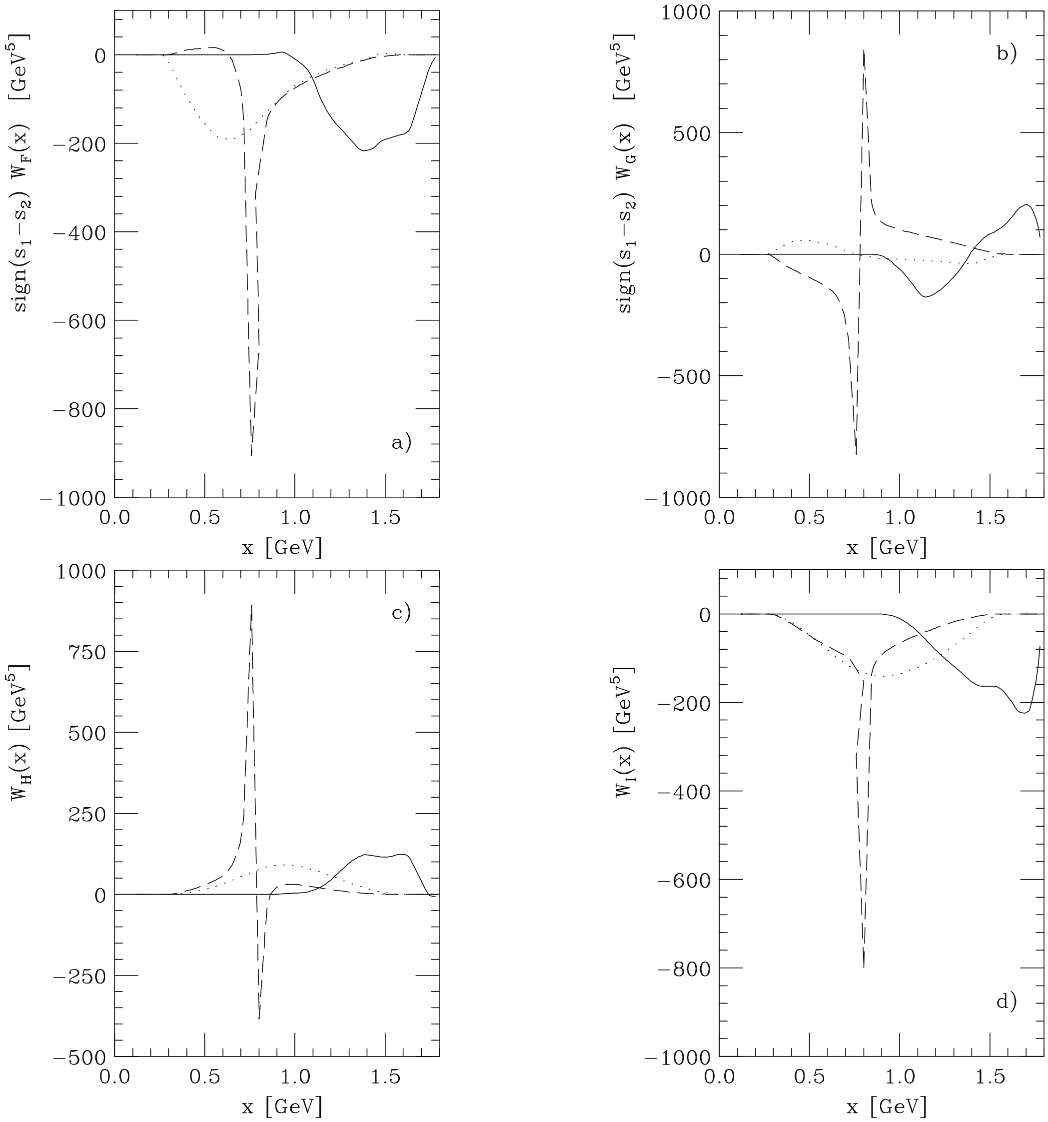

Fig. 5. Invariant mass distributions $x = \sqrt{Q^2} = m(\pi^-\pi^-\pi^+)$ (*solid*), $x = \sqrt{s_{1,2}} = m(\pi^+\pi^-)$ (*dashed*) and $x = \sqrt{s_3} = m(\pi^-\pi^-)$ (*dotted*) of the structure functions $sign(s_1 - s_2)W_F$ **a**, $sign(s_1 - s_2)W_G$ **b**, W_H **c**, W_I **d** in the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ decay mode

structure function W_B. The $(\pi^-\pi^-\pi^+)$ mass distribution (solid line) is dominated by the two higher radial excitations $\rho'(1465)$ and $\rho'(1700)$ of the ρ resonance in (20). The narrow peak in the $\sqrt{s_1} = \sqrt{s_2} = m(\pi^+\pi^-)$ distribution (dashed line) shows the dominance of the ω subresonance in the vector current. The shape of the $\sqrt{s_3}$ distribution (dotted line) is fixed by phase space restrictions and the $\sqrt{Q^2}$ and $\sqrt{s_1}, \sqrt{s_2}$ distributions through $s_3 = Q^2 - s_1 - s_2 + 3m_\pi^2$. A comparison of W_B in Fig. 4 with W_A in Fig. 1 shows that the contribution to the decay rate from W_B is small compared to the axial vector structure function W_A. In fact, using (12) we find that W_B contributes numerically 0.4% to the decay rate, which is slightly below the branching fraction that has been reported by ARGUS [12], namely 0.6%. Due to the large uncertainties in the axial vector part, in particular in the a_1 width, isospin violating effects cannot be seen by a rate measurement. One could try to disentangle the structure functions W_A and W_B by analyzing the difference in the $\cos\beta$ distribution (see [10,11]) (β denotes the angle between the normal of the three pion plane and the direction of the laboratory in the hadronic rest frame). However, the sensitivity to the difference in the β distribution for these two structure functions is fairly small [16] and such an analysis is probably not possible with the current statistics.

Much more promising is an analysis of the vector current contribution through the measurement of the interference effects between the vector current contribution with the dominating axial vector current contribution, *i.e.* through a measurement of the structure functions $sign(s_1-s_2)W_F, sign(s_1-s_2)W_G, W_H$ and W_I. The three meson and two meson invariant mass distributions for these structure functions are shown in Fig. 5. The shape of the three pion invariant mass distributions (solid lines) is determined by the interference of the a_1 resonance in the form factors F_1 and F_2 and the $T_\rho^{(4m)}$ resonance in (20, 23). The ρ' and ρ'' peaks are visible in all four structure functions. Similarly, the narrow peaks around 800 MeV in the $m(\pi^+\pi^-)$ invariant mass distribution is a consequence of the interfering ρ resonance in F_1 and F_2 with the product of $\mathrm{BW}_\omega(s_{1,2})\mathrm{BW}_\rho(s_{1,2})$ in F_3 as described in (31). The structure functions $sign(s_1 - s_2)W_F$ and W_H are the most promising candidates to extract a vector current contribution in an unambiguously way. An additional scalar contribution of the same size as the vector current contribution discussed before would contribute to two additional structure functions whose angular coefficients are similar to those of W_G and W_I [10], and thus a separation of the vector current and such scalar effects might be very difficult with presently available statistics in the data (see also [16]). On the other hand, the angular distributions which determine the structure functions $sign(s_1 - s_2)W_F$ and W_H differ considerably from those originating from possible spin-zero-spin-one interference effects. Any nonvanishing contribution to these structure functions would therefore be a clear signal of isospin violation.

In the decay $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ we find the effects of isospin violation to be negligibly small. Indeed, the contribution to the decay rate from W_B is of the the order 10^{-3}%. Those the amplitudes in the invariant mass distributions of W_B are very small when compared to the corresponding invariant mass spectra in the decay $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$. Even the distributions in the interference terms do not have significant amplitudes. We therefore conclude that isospin violation in the decay $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ can hardly be measured in presently available data.

To summarize: An isospin violating vector form factor is expected to give a contribution of about 0.4% to the decay rate in the $\tau^- \to \pi^-\pi^-\pi^+\nu_\tau$ decay mode. Sizable interference effects of this vector form factor with the dominating axial vector form factors are discussed in detail. These effects could be observed with presently available statistics without reconstructing the τ rest frame. Any nonvanishing contribution to the corresponding structure functions would be a clear signal of isospin violation. The corresponding signals in the $\tau^- \to \pi^0\pi^0\pi^-\nu_\tau$ decay are found to be considerably smaller.

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